EXISTENCE AND UNIQUENESS SOLUTIONS OF NON LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AND APPLICATIONS

A proposed synopsis submitted to

Swami Ramanand Teeth Marathwada University, Nanded

For the award of

Doctor of Philosophy

In Mathematics under the faculty of Science and Technology

By

Rahatikar Shashank Venkatrao

N.E.S Science College,Shree Nagar,Nanded (M.S)Emaill

:shashankrahatikar@gmail.com

Under the Supervision of

Dr.V.C. Borkar

M.sc, Ph.D

Department of Mathematics &

StatisticsYeshwant Mahavidyalaya

Nanded

Dist. Nanded (M.S)

EXISTENCE AND UNIQUENESS SOLUTIONS OF NON LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AND APPLICATIONS

Introduction:

The history of fractional derivative begins in 1695 when L'hospital asked Leibniz that what meaning could be ascribed to $D^n f$ if n=1/2 in his letter, Leibniz replied it would be a paradox. after that the subject of fractional calculus caught the attention of other great mathematicians, many of whom directly or indirectly contributed to its development.They included Euler, Laplace, Fourier, Lacroix, Abel, Riemann and Liouville. Lacroix was first mathematician who released paper mentioning fractional derivatives in it. He began with the polynomial $f(x)=x^m$, where m is a positive integer, and differentiated it n times where $m \geq n$ to get $\frac{Df}{Dx}=\frac{m!}{(m-n)!}$,then he used Legendres symbol Γ to have $\frac{D^n f}{Dx^n}.=\frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{(m-n)}$ Using this formula when m=1 and $n=\frac{1}{2}$ he obtained $d^{\frac{1}{2}}f = \frac{2\sqrt{x}}{\sqrt{\pi}}$

In the middle of the nineteenth century, fractional calculus was created as a mathematical theory. Period of the nineteenth century (Ross, 1977) it has been proved that many things have changed in the last 100 years. Fractional differential equations can be used to more precisely model physical systems rather than conventional differential equations compared to the traditional differential equation.

Most fractional differential equations (FDEs) don't have exact solution, so approximate and numerical techniques must be used currently several different methods are used to solve fractional differential equations i.e. Adomian decomposition method, Homotopy Perturbation Method, Differential transform method, Series solution for fractional derivative, Conformable fractional derivative and Variation iteration method which gives approximate and exact solutions for linear and nonlinear fractional differential equations.

Since three centuries, fractional calculus became the traditional calculus but not very common amongst science and engineering community, and other areas of sciences can be successfully acclimatized by the use of fractional derivatives

.

There are three basic definitions of fractional derivatives based on which The fractional Calculus and Fractional differential equation are developed. These are as

a) Riemann Liouville Fractional Derivative:

For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^{\alpha} \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx$$

b) Caputo's Fractional Derivative:

For $\alpha \in [n-1, n)$, the α-derivativeof f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^{\alpha} \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx$$

c) Mittag-Leffler Function:

The one and two-parameter Mittag-Leffler functions aredefined, respectively, by:

$$\mathrm{E_a}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\mathrm{an}+1)}, \qquad \mathrm{a}>0$$

$$\mathrm{E_{a,b}}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\mathrm{an}+\mathrm{b})} \qquad \mathrm{a}>0, \quad \mathrm{b}>0$$

Some Methods to Solve Fractional Differential Equations:

1. Laplace Transform Method:

The Laplace transformation method can be used to solve Fractional Differential Equations. The Laplace transform of Fractional derivative is given by,

a) Laplace Transform of Riemann Liouville Fractional Derivative

$$L\{ {}_0D_t^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{p-k-1} f(t) \right]_{t=0},$$

where $(n-1 < p \leq n)$

b) Laplace Transform of Caputo Fractional Derivative

$$L\{{}_0^C D_t^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0),$$

where $(n-1 < p \leq n)$.

2. Higher order Numerical methods

There are two approaches to solve these problems, The first approach is based on a direct discretisation of the fractional differential operator. we obtain a numerical method for solving a linear fractional differential equation with order O<a<1. The order of convergence of the numerical method is $o(h^{(3-a)})$, the second approach is based on discretisation of the integral form of the frac1ional differential equation and weobtain a fractional Adams type method for a non linear fractional differential equation of any order a>0.

3 . The Legendre wavelet method:

An operational matrix of fractional order integration is derived and isutilized to reduce the fractional differential equations to system of algebraic equations.

4. Homotopy Analysis Method:

A homotopy [5] between two continuous function $f(x)$ and $g(x)$ from a topological space X to a topological space Y is formally defined to be a continuous function $H: X \times [0,1]$ Y from the product of the space X with the unit interval[0,1] to Y such that, if x in X then $H(x;0) = f(x)$ and $H(x;1) = f(x)$.

5. The second kind Chebyshev wavelet method:

The second kind Chebyshev wavelet method is presented for solving linear and nonlinear fractional differential equations. We first construct the second kind Chebyshev wavelet and then derive the operational matrix of fractional order integration. The operational matrix of fractional order integration is utilized to reduce the fractional differential equations to system of algebraic equations.

Review of literature and recent developments in the subject area:

In 2022, Vidushi Gupta, Natrajan Valiammal, Chokkalingam Ravichandrean, Kottakaran Sooppy Nisar, Fahd Jarad **[3]** have studied the investigation of existence and uniqueness of impulsive dynamical fractional systems with quadratic perturbation of second type subject to non local boundary conditions is presented and proved. By employing the fractional theory, Banach contraction technique, and Krasnoselskii's fixed point theorem, they derived some sufficient conditions to ensure the existence of the system.

In 2020, Sunil Kumar, Amit Kumar, Syed Abbas, Maysaa Al Qurashi, Dumitru Baleanu**[6**] has mainly explores and applies a modified form of the analytical method, namely the homotopy analysis transform method (HATM) for solving time-fractional Cauchy reaction-diffusion equations (TFCRDEs). Then mainly they address the error norms L_2 and L_∞ for aconvergence study of the proposed method. They also find existence, uniqueness and convergence in the analysis for TFCRDEs. The projected method is illustrated by solving some numerical examples. The obtained numerical solutions by the HATM method show that it is simple to employ. An excellent conformity obtained between the solution got by the HATM method and the various well-known results available in the current literature. Also the existence and uniqueness of the solution have been demonstrated

In 2019. Dnyanoba Dhaigude, Bakr Rizqan **[12]** develop a monotone iterative technique by introducing upper and lower solutions to Riemann- Liouville fractional differential equations with deviating arguments and integral boundary conditions. As an application of this technique, existence and uniqueness results are obtained.

In 2018 Hossein Fazli, Juan J Nieto, Fariba Bahrami **[5]** have proved the the existence, uniqueness and smoothness of solutions for differential equations with Riemann-Liouville sequential fractional derivatives admitting only the existence of a lower solution. The appropriate fractional Sobolev spaces equipped with a suitable partial order are introduced, and applying some fixed point theorems the results are proved. The approximate solutions are presented to the corresponding initial value problems

In 2017, Mujeeb ur Rehman, Amna Idrees, Umar Saeed **[8]** have studied a numerical method obtain approximate solutions for a certain class of fractional differential equations. The method reduces the underlying differential equation to system of algebraic equations. An algorithm is presented to compute the coefficient matrix for the resulting algebraic system. Several examples with numerical simulations are provided to illustrate effectiveness of the method.

.

In 2015, Vasile Lupulescu, Le Si Dong,Ngo Van Hoa **[2]** have studied the random fuzzy fractional integral and differential equations and Under Lipschitz condition they obtain the existence and uniqueness theorems of solutions for two general forms of random fuzzy fractional integral equations. To prove this assertion they use an idea of successive approximations. Moreover, the approach is followed to prove the existence and uniqueness of solutions for random fuzzy fractional initial value problem under Caputo- type fuzzy fractional derivatives

In 2014,Pholakrit Thiramaus, Sotiris K Ntouyas, Jessada Tariboon **[7]** have studied the existence and uniqueness of solutions for a fractional boundary value problem involving Hadamard-type fractional differential equations and non local fractional integral boundary conditions. The results are based on some classical fixed point theorems.

In 2013 Rahmat Darzi, B Mohammadzadeh, Abdolali Neamaty, Dumitru Baleanu **[20]** have investigate the existence and uniqueness of solution for fractional boundary value problem for nonlinear fractional differential equation with the integral boundary conditions where denotes Caputo derivative of order by using the fixed point theory. They apply the contraction mapping principle and Krasnoselskii's fixed point theorem to obtain some new existence anduniqueness results

In 2013, Bin Zheng, Chuanbao Wen **[9]** have proposed a new fractional sub-equation method for finding exact solutions of fractional partial differential equations (FPDEs) in the sense of modified Riemann-Liouville derivative, which is the fractional version of the known (G'/G) method. To illustrate the validity of this method, they applied it to7 the space-time fractional Fokas equation, the space-time fractional -dimensional dispersive long wave equations and the space-time fractional fifth- order Sawada-Kotera equation. As a result, some new exact solutions for them are successfully established

In 2011 Bashir Ahmad, Juan J Nieto, Ahmed Alsaedi **[4]** have studied study a boundary value problem of nonlinear fractional differential equations of order q $(1< q \leq 2)$ with non- separated integral boundary conditions. Some new existence and uniqueness results are obtained by using some standard fixed point theorems and Leray-Schauder degree theory.

In 2010, Bashir Ahmad, Ahmed Alsaedi**[13]** solved some initial value problem for a coupled Caputo type nonlinear fractional differential system of higher order. As a first problem,they exhibited that the non homogeneous terms in the coupled fractional differential system depend on the fractional derivatives of lower orders only. Then the non homogeneous terms in the fractional differential system are allowed to depend on the unknown functions together with the fractional derivative of lower orders. their method of analysis is based on the reduction of the given

system to an equivalent system of integral equations. Applying the nonlinear alternative of Leray-Schauder, they prove the existence of solutions of the fractional differential system. The uniqueness of solutions of the fractional differential system is established by using the Banach contraction principle. some illustrative examples are also given

In 2009 Malik Muslim **[14]** have study a fractional order semilinear differential equation in an arbitrary Banach space X. he used the analytic semigroup theory of linear operators and fixed point method to establish the existence and uniqueness of solutions of the given problem. he also prove the existence of a global solution. Existence and convergence of an approximate solution to the given problem is also proved in a separable Hilbert space.

Objectives of proposed research:

1) To study linear and nonlinear ordinary and partial differential equations

2) To study basic concepts of fractional calculus.

3) To study varius methods to solve fractional differential equations.

4) To study Various methods for Existance and uniqueness of solutions of fractional differential equations.

5) To study and Analysis of development in results for fractional differential equations.

6) To study recent applications of fractional differential equations and its Various methods for Existance and uniqueness of solutions of fractional differential equations in a various fields of science and technology

7) To prove that how mathematics is useful in other branches of science. We hope that our research work will be useful in applied sciences and engineering.

8) Hence, in the present work, we will try to develop the new analytical results and new iterative techniques for fractional differential equation and to obtain their solution. These will represent graphically by Mathematica, MATLAB, Python, Scilab and Mathcad.

Research Methodology:

The following steps will be including in the methodology:

1) To refer the relevant reference books on fractional differential equations.

2) To study Mathematica, MATLAB, Python, Scilab, Mathcad softwares.

3) To study Fractional Calculus

4) To study existence and uniqueness of fractional differential equations

5) To discuss the stability and convergence of various methods.

6) To represent solutions of fractional differential equations graphically by Mathematica, MATLAB and discuss their nature.

Importance of study:

Durying the last decade,fractional calculus has been applied to almost every field. In Physics, Engineering, Biology, and Economics, fractional calculus is becoming a valuable and important tool for analysing scientific problems. Many application of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics. Fractional calculus, which involves the integration and differentiation of an arbitrary or fractional order, provides additional tools that extend the descriptive capacity of calculus beyond the classic integer-order concepts of rate of change and area under the curve.

International Status:

Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibtniz, Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Richtmeyer, Morton, Smith, Anderson, Roache, Lynn R. Willams (1979) Nguyen Huu Denn (1993), Hossein Fazli, Juan J Nieto, Fariba Bahrami

National Status:

Indian's also contributed remarkable in the development of fractional calculus like R. K. Sexana, V.Srivastava, K. N. Rai, S. Das, K. Gupta, S. Saha Roy, V. B. L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna, Aalam, Suneel Kumar,Sunny Chauhan and Pant. S. P.

Acharya worked on some results on fixed point in uniform space. S.L. Sing from Haridwar, R. P. Pant from Nainital, D.N Chaliszar from Gujrat, Vidushi Gupta, Fahd Jarad, Natarajan Valliammal, Chokkalingam Ravichandran, Kottakkaran Sooppy Nisar and many more from India had worked on fixed point theorems and its various applications.

Regional Status:

Many researchers from Maharashtra have been actively involved in the study of fractional calculus. Like M. Kolwankar, A. K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji, K.C. Takale , J. Verma, D. B. Dhaigude.

Scope of research work:

1**)** In recent years, there has been a great deal of interest in fractional differential equations. During the last decade, fractional calculus has been applied to almost every field of Science, Engineering and Mathematics. Many application of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics

2) There are few different methods for obtaining the solution of fractional differential equation (FDE), therefore, we develop new different methods to obtain their solutions and discuss the stability and convergence of the numerical methods.

3) Even many Transformations and their applications has been developed by researchers for solving many problems of science and technology, but an easy, suitable and number of alternate approaches is desired to solve different problems of different domains, so there is a good scope of development in this area.

.

Expected outcomes:

1. We will try to study the development in various methods to solve fractional differential equations.
2. We will try to analyze various methods for existence and uniqueness of solutions of fractional differential equations

3. To find the approximate solutions of fractional differential equations and it will represente graphically by some mathematical software

Purpose of carrying out proposed research work:

The purpose of this research work is to study various methods for existence and uniqueness of solutions of fractional differential equations also we will try to obtain the approximate solution of fractional differential differential equations graphically using varius mathematical softwares.

Plan of work:

Sr. No.	Period	Plan of work	Targets to be Achieved
1	First year	To refer the reference books on differential equation and fractional differential equation to develope different numeric methods for differential equations	Excellent understanding of the subject.
2	Second year	To study and analize the number of reputed research papers in fractional calculus.	try to obtain new results based on existing work and submit some papers.
3	Third year	Publishing some papers and writing of the thesis.	Some more papers and thesis submission.

References:

[1] I. Podlubny Fractional Differential Equations. Academics press, New York, 1999.

[2] Vasile Lupulescu,Le Si Dong Ngo Van hoa Existence and uniqueness of solutions for random fuzzy fractional integral and differential equations *Journal of Intelligent & Fuzzy* Systems *29 (1), 27-42,2015*

[3] Vidushi Gupta, Fahd Jarad, Natarajan Valliammal, Chokkalingam Ravichandran, Kottakkaran Sooppy Nisar, Existence and uniqueness of solutions for fractional nonlinear hybrid impulsive system *Numerical Methods for Partial Differential Equations 38 (3), 359-371, 2022*

[4] Bashir Ahmad, Juan J Nieto, Ahmed Alsaedi Existence and uniqueness of solutions for nonlinear fractional differential equations with non- separated type integral boundary conditions *Acta Mathematica Scientia 31 (6),2122-2130, 2011*

[5] Hossein Fazli, Juan J Nieto, Fariba Bahrami On the existence and uniqueness results for nonlinear sequential fractional differential equations *Appl. Comput. Math 17 (1),36-47, 2018*

[6] Sunil Kumar, Amit Kumar, Syed Abbas, Maysaa Al Qurashi, Dumitru Baleanu A modified analytical approach with existence and uniqueness for fractional Cauchy reaction- diffusion equations *Advances in Difference Equations 2020 (1), 1-18,2020*

[7] Phollakrit Thiramanus, Sotiris K Ntouyas, Jessada Tariboon Existence and uniqueness results for Hadamard-type fractional differential equations with nonlocal fractional integral boundary conditions *Abstract and Applied Analysis 2014*

[8] Mujeeb ur Rehman, Amna Idrees, Umer Saeed A quadrature method for numerical solutions of fractional differential equations *Applied Mathematics and Computation 307,38-49, 2017*

[9] Bin Zheng, Chuanbao Wen Exact solutions for fractional partial differential equations by a new fractional sub-equation method *Advances in Difference Equations 2013 (1), 1-12,2013*

[10] P. Igor, Fractional differential equations *New York: Academics Press*, 1999.

[11] An Introduction to the Fractional Calculus and Fraction Differential Equation By Kenneth Miller Bertram Ross

[12] Dnyanoba Dhaigude, Bakr Rizqan Existence and uniqueness of solutions of fractional differential equations with deviating arguments under integral boundary conditions *Kyungpook Mathematical Journal* 59 *(1), 191-202, 2019h=4*

[13]Bashir Ahmad, Ahmed Alsaedi Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fractional differential equations *Fixed Point Theory and Applications 2010, 1-17, 2010*

[14] Malik Muslim Existence and approximation of solutions to fractional differential equations *Mathematical and Computer Modelling 49 (5-6), 1164-1172, 2009*

[15] Taige Wang, Feng Xiefractional differential equations with integral boundary conditions *The Journal of Nonlinear Sciences and its Applications 1 (4), 206-212, 2008*

[16] Christopher S GoodrichExistence and uniqueness of solutions to a fractional difference equation with nonlocal conditions *Computers & Mathematics with Applications61 (2), 191-202, 2011*

[17] Dumitru Baleanu, Shahram Rezapour, Hakimeh Mohammadi Some existence results on nonlinear fractional differential equations *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 371 (1990), 20120144, 2013*

[18] F. Mainardi, "Fractional Calculus: Theory and Applications," *Mathematics*, vol. 46, pp. 4, 2018.

[19] D. K. J. S. Sumit Gupta, "Numerical study for systems of fractional differential equations via Laplace transform," *Journal of the Egyptian Mathematical Society,* vol. 23 , no. 2, pp. 256-262, 2015.

[20] Rahmat Darzi, B Mohammadzadeh, Abdolali Neamaty, Dumitru Baleanu On the existence and uniqueness of solution of a nonlinear fractional differential equations *J. Comput. Anal. Appl 15 (1), 152-162, 2013*

Research Scholar Research Supervisor